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FINITE ELEMENT SOLUTION OF THE NAVIER-STOKES EQUATIONS USING A SUPG FORMULATION.

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1 INTRODUCTION

This paper shows the results obtained in the resolution of the Navier-Stokes equations in a two-dimensional domain for given initial and boundary conditions. The consideration of flows with large enough Reynolds numbers, causes a certain amount of numerical instability in the resolution of the system of equations, when using the standard Galerkin formulation. Although this may be suppressed by a severe refinement in the mesh, the usage of a SUPG (Streamline Upwinding / Petrov-Galerkin) algorithm, manages to overcome this difficulty with less computational cost. The SUPG algorithm, leads the flow in the suitable streamline direction by adding an artificial diffusion term, and also makes an upwind weighting of the numerical approximation of the flow, thanks to the usage of weight functions different to the trial functions. The numerical instability is thus, drastically reduced even for large convective-terms-including equations.

2 FUNDAMENTAL EQUATION. FINITE ELEMENT FORMULATION.

The governing equations for the unsteady, incompressible, viscous flow are the following:

$$u_i + u_j u_{i,j} - \boldsymbol{n}\left(u_{i,j} + u_{j,i}\right)_j + \frac{1}{\boldsymbol{r}} p_{,i} = f_i \qquad u_{i,i} = 0, \tag{1}$$

with the initial and boundary conditions, $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})$ in Ω and $\mathbf{u} = \mathbf{g}$ on $¶\Omega \times (0,T)$.
Multiplying the equation by a weighting function, integrating over the domain and integrating by parts, we arrive to the weak form of the Navier-Stokes equations:

$$\int_\Omega w_i\left(\dot{u}_i + u_j u_{i,j,j} - f_i\right) + w_{i,j}\boldsymbol{n}\left(u_{i,j} + u_{j,i}\right) - w_{i,i}\frac{1}{\boldsymbol{r}} p \; d\Omega = 0 \quad \int_\Omega u_{i,i}\, q \, d\Omega = 0, \tag{2}$$

adding the perturbation term $\tilde{p}_i^h$ to the weighting function and discretizating $\mathbf{u}$ and p in terms of the Q1P0 basic element $[\mathbf{u}(\mathbf{x}) \approx \mathbf{u}_h(\mathbf{x}) = \sum_{n=1}^{N} \mathbf{u}_n \boldsymbol{f}_n(\mathbf{x}), \;\; p(\mathbf{x}) = p_h(\mathbf{x}) = \sum_{m=1}^{M} p_m \boldsymbol{c}_m(\mathbf{x})]$, the former equation turns into:

$$\int_\Omega w_i^h\left(\dot{u}_i^h + u_j^h u_{i,j}^h - f_i\right) + w_{i,j}\boldsymbol{n}\left(u_{i,j} + u_{j,i}\right) - w_{i,i}\frac{1}{\boldsymbol{r}} p \; d\Omega \quad +$$

$$+ \sum_{e=1}^{N_e} \int_{\Omega_e} \tilde{p}_i^h \left(u_i^h + u_j^h u_{i,j}^h - \boldsymbol{n}\left(u_{i,j}^h + u_{j,i}^h\right) + \frac{1}{\boldsymbol{r}} p_{,i}^h - f_i \right) d\Omega_e = 0 \qquad \int_\Omega u_{i,i}^h q^h \, d\Omega = 0, \tag{3}$$

where $\tilde{p}_i^h$, is the streamline upwind contribution to the weighting function and can be written as:

$$\tilde{p}_i^h = c_j w_{i,j}^h \qquad c_i = \frac{z h_e u_i}{2|u_e|} \qquad \text{with} \qquad z = \mathbf{coth}(Pe) - \frac{1}{Pe} \approx min\left[1, \frac{1}{Pe}\right] \qquad Pe = \frac{|u_e| h_e}{2n}, \qquad (4)$$

being u_e the velocity in the basic element centre, u_i the i-component of the velocity in the element centre and h_e the characteristic length of the basic element.
The equation (2) can then be expressed in matrix form as:

$$\mathbf{M}\,\dot{\mathbf{u}} + \mathbf{c}(\mathbf{u}) + n\mathbf{A}\mathbf{u} + \mathbf{B}\mathbf{p} = \mathbf{f} \qquad\qquad \mathbf{B}^T\mathbf{u} = \mathbf{0}, \qquad (5)$$

system of differential non-linear equations that will be turned into a linear one, using a successive approximation algorithm, this is, approximating the convective term as follows:

$$\mathbf{c}(\mathbf{u}_k) \approx \mathbf{C}(\mathbf{u}_{k-1})\,\mathbf{u}_k = \int_\Omega (u_{k-1})_j^h (u_k)_{i,j}^h \cdot w_i^h \, d\Omega + \sum_{e=1}^{N_e} \int_{\Omega_e} (u_{k-1})_j^h (u_k)_{i,j}^h \cdot \tilde{p}_i^h \, d\Omega \qquad (6)$$

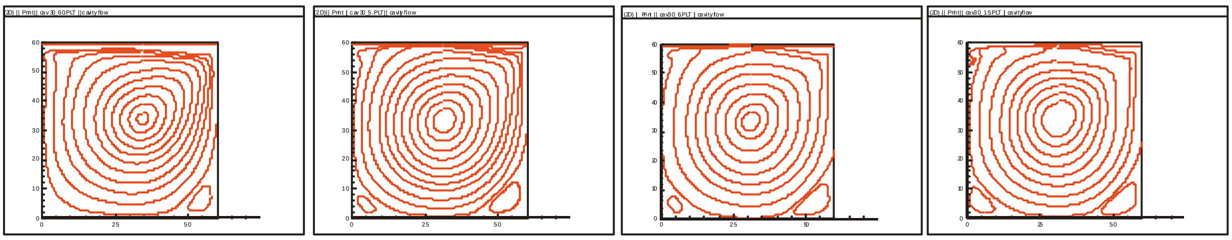
Finally the temporal integration is made in terms of an implicit backward scheme:

$$\mathbf{M}\left(\frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t}\right) + \mathbf{C}\mathbf{u}_{n+1} + n\mathbf{A}\mathbf{u}_{n+1} - \mathbf{B}\mathbf{p}_{n+1} = \mathbf{f}_{n+1} \qquad \mathbf{B}^T\mathbf{u}_{n+1} = \mathbf{0} \qquad (7)$$

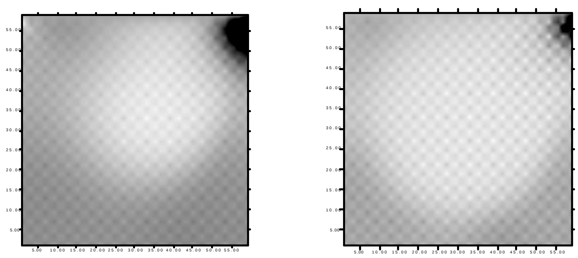
System which is solved making use of a Crout algorithm, with a Sky-line loading procedure for the matrices involved. The integration process is made based upon a two-point Gauss numerical integration over the basic elements.

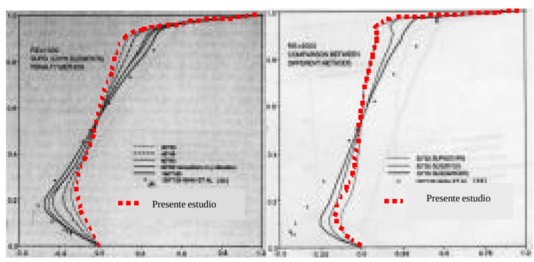
3 NUMERICAL EVALUATION AND CONCLUSIONS

The program is checked with the problem of the flow in a square cavity for Reynolds numbers of 1.000, 5.000, 10.000 y 15.000 on a 31x31 node regular mesh, and the streamlines obtained for the velocity field can be seen bellow:



The results for the pressure field are sketched (with no smoothing) for Reynolds numbers of 1.000 and 10.000.





The present algorithm solves the Navier-Stokes equations with results similar to those obtained by other authors, for the same mesh refinement. This can be observed in the last two graphs representing the horizontal components of the velocity along a central vertical line of the square domain, for different formulations and rate of refinement.